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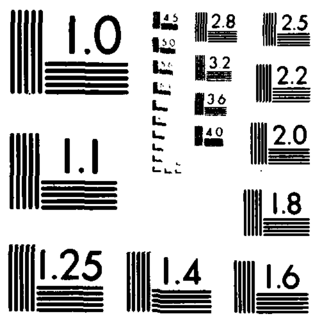

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A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes

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ENGINEERING & INDUSTRIAL RESEARCH STATION
Department of Aerospace Engineering

by

Z. U. A. Warsi

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Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies is presented. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when equations depending on all three independent variables are considered. The method also allows, in a straight forward manner the possibility of coordinate contraction in the desired regions.

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A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes*

by

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Abstract

Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies is presented. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when equations depending on all three independent variables are considered. The method also allows, in a straight forward manner, the possibility of coordinate contraction in the desired regions.

An exact solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere with coordinate contraction is presented to demonstrate that by using these equations it is possible to generate three-dimensional coordinates between analytically specified surfaces of simple forms by analytical means.

The fundamental constraining equations which have been adopted for the generation of coordinates are $\Delta_2\xi = 0$ and $\Delta_2\eta = 0$, where Δ_2 is the surface Beltrami operator of the second order.

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†Professor

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1. Introduction

At present a number of techniques are under active development for the generation of three-dimensional body-oriented coordinate systems for use in the numerical solution of the Navier-Stokes equations and other field equations where the exact specification of the boundary conditions is of prime importance. Among these efforts two easily discernable groups can be formed, (i) algebraic methods, and (ii) the elliptic equations method. In the first group the grid points in space are obtained by some interpolation or blending functions scheme which depends on the given boundary data. The choice of the interpolation scheme or of the blending functions is crucial in achieving a desired order of smoothness and distribution of the grid points in space. This line of effort has actively been considered by Eiseman [1,2], Smith and Weigel [3], and Eriksson [4]. In the second group of efforts, a set of three poisson equations in the curvilinear coordinates are first inverted and then solved for the Cartesian coordinates under the prescribed values at the given boundaries. Thus in essence all the methods of the second group are a straight forward extension of the work of Thompson et al [5] in two dimensions. Research in this area has been conducted by Mastin et al [6], Yu [7], Ghia et al [8], and Graves [9].

At this stage of research it is premature to compare the two groups since neither of them have been fully investigated for their inherent potentials. However, based on the success of the differential equations approach in two dimensions, e.g. [5], it is desirable to further investigate the elliptic equations approach for the generation of coordinates.

The elliptic equations approach presented in this paper is different from the approaches adopted in the previously cited works, i.e., References

[6] - [9]. The proposed method depends heavily on the formulae of Gauss and on the concept of principal curvatures of a surface. It has been shown that a fruitful arrangement of the classical differential-geometric results can yield a method which is easily programmable on a computing machine, and which at any time solves a two-dimensional partial differential equation of the form used in Ref. [5]. In this paper only the theoretical development of the method along with a technique to redistribute the coordinate surfaces near the inner boundary surface has been considered. The developed equations have been solved for the generation of three-dimensional coordinates between an inner prolate ellipsoid and an outer spherical surface in an exact analytic form.

2. Notation and Collection of Formulas

In what follows, the general coordinates are denoted as x^i ($i = 1, 2, 3$). However when an expression has been expanded out in full and there is no use for an index notation, we have set

$$x^1 = \xi \ , \ x^2 = \eta \ , \ x^3 = \zeta \ .$$

The derivatives of the position vector $\underset{\sim}{r} = (x,y,z)$ are denoted as

$$\underset{\sim}{r}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i} \ , \ \underset{\sim}{r}_{ij} = \frac{\partial^2 \underset{\sim}{r}}{\partial x^i \partial x^j} \ .$$

The covariant components of the metric tensor are

$$g_{ij} = \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j \tag{2.1}$$

while the contravariant components are given by

$$g^{ij} g_{kj} = \delta^i_k \tag{2.2}$$

Thus in three dimensions

$$g = \det(g_{ij})$$

$$= g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} \tag{2.3}$$

Writing

$$G_1 = g_{22}g_{33} - (g_{23})^2$$

$$G_2 = g_{11}g_{33} - (g_{13})^2$$

$$G_3 = g_{11}g_{22} - (g_{12})^2$$

$$G_4 = g_{13}g_{23} - g_{12}g_{33}$$

$$G_5 = g_{12}g_{23} - g_{13}g_{22}$$

$$G_6 = g_{12}g_{13} - g_{11}g_{23} \tag{2.4}$$

we have

$$g^{11} = G_1/g \ , \ g^{22} = G_2/g \ , \ g^{33} = G_3/g$$

$$g^{12} = G_4/g \ , \ g^{13} = G_5/g \ , \ g^{23} = G_6/g \tag{2.5}$$

The Christoffel symbols based on the metric g_{ij} are

$$\Gamma^i_{jk} = g^{i\ell}[jk,\ell]$$

where

$$[jk,\ell] = \underset{\sim}{r}_{jk} \cdot \underset{\sim}{r}_\ell \tag{2.6}$$

$$= \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right)$$

and repeated lower and upper indices imply summation. In the sequel we have also used the surface Christoffel symbols which have been denoted as $T^\alpha_{\beta\gamma}$, where the Greek indices range over (1,2) or (3,1) or (2,3).

The coordinate which is held fixed to account for the surface geometry is denoted by a superscript in parentheses. Thus the unit normal vector on the surface ν = const. is given by

$$\underset{\sim}{n}^{(\nu)} = (\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta)/|\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta| \tag{2.7}$$

where

$$\left.\begin{array}{ll} \nu = 1 : \alpha = 2 \ , \ \beta = 3 & (\text{surface } x^1 = \text{const.}) \\ \nu = 2 : \alpha = 3 \ , \ \beta = 1 & (\text{surface } x^2 = \text{const.}) \\ \nu = 3 : \alpha = 1 \ , \ \beta = 2 & (\text{surface } x^3 = \text{const.}) \end{array}\right\} \tag{2.8}$$

The rectangular Cartesian components of $\underset{\sim}{n}^{(\nu)}$ are denoted as

$$\underset{\sim}{n}^{(\nu)} = (X^{(\nu)} \ , \ Y^{(\nu)} \ , \ Z^{(\nu)}) \tag{2.9}$$

The coefficients of the second fundamental form are denoted by $S^{(\nu)}$, $T^{(\nu)}$ and $U^{(\nu)}$ defined as

$$\left.\begin{array}{l} S^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\alpha} \quad (\text{no sum on } \alpha) \\ T^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\beta} \\ U^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\beta\beta} \quad (\text{no sum on } \beta) \end{array}\right\} \tag{2.10}$$

where (ν,α,β) are in the cyclic permutations of $(1,2,3)$, in this order.

The partial derivatives of the second order are expressible in terms of the first order as

$$\underset{\sim}{r}_{ij} = \Gamma^k_{ij}\underset{\sim}{r}_k \tag{2.11}$$

For a surface on which one of the coordinates is fixed, the Gauss' equations are

$$
\left.
\begin{aligned}
\underset{\sim}{r}_{\alpha\alpha} &= T^{\gamma}_{\alpha\alpha}\underset{\sim}{r}_{\gamma} + S^{(\nu)}\underset{\sim}{n}^{(\nu)} \\
\underset{\sim}{r}_{\alpha\beta} &= T^{\gamma}_{\alpha\beta}\underset{\sim}{r}_{\gamma} + T^{(\nu)}\underset{\sim}{n}^{(\nu)} \\
\underset{\sim}{r}_{\beta\beta} &= T^{\gamma}_{\beta\beta}\underset{\sim}{r}_{\gamma} + U^{(\nu)}\underset{\sim}{n}^{(\nu)}
\end{aligned}
\right\} \tag{2.12}
$$

Where (ν,α,β) are in the permutational sequences of (1,2,3) as shown in (2.8), and the repeated index γ implies summation on the two indices of a surface.

The sum of the principal curvatures of the surface ν = const. is, [10],

$$
k_1^{(\nu)} + k_2^{(\nu)} = (g_{\alpha\alpha}U^{(\nu)} - 2g_{\alpha\beta}T^{(\nu)} + g_{\beta\beta}S^{(\nu)})/G_{\nu} \tag{2.13}
$$

where in writing equation (2.13) for a particular value of ν, use must be made of Eqs. (2.8) and (2.10). We now introduce two second order surface differential operators by using (2.8), which for ν = const. are

$$
D^{(\nu)} \equiv g_{\beta\beta}\partial_{\alpha\alpha} - 2g_{\alpha\beta}\partial_{\alpha\beta} + g_{\alpha\alpha}\partial_{\beta\beta} \tag{2.14}
$$

$$
\begin{aligned}
\Delta_2^{(\nu)} \equiv \frac{1}{\sqrt{G_{\nu}}}[\partial_{\alpha}\{\frac{1}{\sqrt{G_{\nu}}}(g_{\beta\beta}\partial_{\alpha} - g_{\alpha\beta}\partial_{\beta})\} \\
+ \partial_{\beta}\{\frac{1}{\sqrt{G_{\nu}}}(g_{\alpha\alpha}\partial_{\beta} - g_{\alpha\beta}\partial_{\alpha})\}]
\end{aligned} \tag{2.15}
$$

As is well known, the operator Δ_2 is the Beltrami differential operator of the second order [10].

The three space Christoffel symbols which have been referred to in the next section are given below.

$$\Gamma^3_{11} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \xi} + G_6(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta})]$$

$$+ \frac{G_3}{g}(x_{\xi\xi}x_\zeta + y_{\xi\xi}y_\zeta + z_{\xi\xi}z_\zeta) \qquad (2.16)$$

$$\Gamma^3_{12} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \eta} + G_6 \frac{\partial g_{22}}{\partial \xi}]$$

$$+ \frac{G_3}{g}(x_{\xi\eta}x_\zeta + y_{\xi\eta}y_\zeta + z_{\xi\eta}z_\zeta) \qquad (2.17)$$

$$\Gamma^3_{22} = \frac{1}{2g}[G_5(2 \frac{\partial g_{11}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_6 \frac{\partial g_{22}}{\partial \eta}]$$

$$+ \frac{G_3}{g}(x_{\eta\eta}x_\zeta + y_{\eta\eta}y_\zeta + z_{\eta\eta}z_\zeta) \qquad (2.18)$$

3. Formulation of the Problem

The principal idea of the method to be presented is to generate a series of surfaces on each of which a certain a'priori chosen variable or coordinate is kept fixed. Each surface to be generated starts from a given curve of the inner body and ends on the corresponding curve of the outer boundary, cf. Fig. 1. A routine, preferably a spline fit, can then be used to join the successive generated surfaces so as to have a smooth three-dimensional computational net for solving other physical field equations.

To illustrate the method, we take $\nu = 3$ or $x^3 = \zeta$ = constant on each surface to be generated. Thus $\alpha = 1$ and $\beta = 2$, viz., α and β respectively correspond to the coordinates $x^1 = \xi$ and $x^2 = \eta$. For the sake of brevity of notation we will not use the superscript (3) unless it becomes necessary. Thus from (2.10)

$$S = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} \; , \; T = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} \; , \; U = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} \tag{3.1}$$

where

$$\underset{\sim}{n} = \underset{\sim}{i}X + \underset{\sim}{j}Y + \underset{\sim}{k}Z \tag{3.2}$$

Equations (2.12) are

$$\underset{\sim}{r}_{\xi\xi} = T^{\gamma}_{11}\underset{\sim}{r}_{\gamma} + S\underset{\sim}{n} \tag{3.3}$$

$$\underset{\sim}{r}_{\xi\eta} = T^{\gamma}_{12}\underset{\sim}{r}_{\gamma} + T\underset{\sim}{n} \tag{3.4}$$

$$\underset{\sim}{r}_{\eta\eta} = T^{\gamma}_{22}\underset{\sim}{r}_{\gamma} + U\underset{\sim}{n} \tag{3.5}$$

From (2.14) and (2.15) the operators $D^{(3)}$ and $\Delta_2^{(3)}$ are

$$D \equiv g_{22} \frac{\partial^2}{\partial\xi^2} - 2g_{12} \frac{\partial^2}{\partial\xi\partial\eta} + g_{11} \frac{\partial^2}{\partial\eta^2} \tag{3.6}$$

$$\Delta_2 \equiv \frac{1}{\sqrt{G_3}}\Big[\frac{\partial}{\partial\xi}\Big\{\frac{1}{\sqrt{G_3}}\Big(g_{22} \frac{\partial}{\partial\xi} - g_{12} \frac{\partial}{\partial\eta}\Big)\Big\}$$

$$+ \frac{\partial}{\partial\eta}\Big\{\frac{1}{\sqrt{G_3}}\Big(g_{11} \frac{\partial}{\partial\eta} - g_{12} \frac{\partial}{\partial\xi}\Big)\Big\}\Big] \tag{3.7}$$

We now multiply equations (3.3) - (3.5) respectively by g_{22}, $-2g_{12}$, g_{11} adding and using equations (2.13) and (2.15) to have

$$D\underset{\sim}{r} + G_3(\underset{\sim}{r}_\xi\Delta_2\xi + \underset{\sim}{r}_\eta\Delta_2\eta) = G_3\underset{\sim}{n}(k_1 + k_2) \tag{3.8}$$

where

$$\left.\begin{aligned} \Delta_2\xi &= \frac{1}{G_3}(2g_{12}T^1_{12} - g_{22}T^1_{11} - g_{11}T^1_{22}) \\ \Delta_2\eta &= \frac{1}{G_3}(2g_{12}T^2_{12} - g_{22}T^2_{11} - g_{11}T^2_{22}) \\ G_3 &= g_{11}g_{22} - (g_{12})^2 \end{aligned}\right\} \tag{3.9}$$

To obtain an expression for $k_1 + k_2$ consider equation (2.11) and utilize the property that $\underset{\sim}{n}$ is orthogonal to $\underset{\sim}{r}_\xi$ and $\underset{\sim}{r}_\eta$, so that

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} = \Gamma^3_{11}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} = \Gamma^3_{12}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} = \Gamma^3_{22}(\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta})$$

(3.10)

where all the derivatives with respect to ζ are evaluated at ζ = constant. Multiplying Eq. (3.8) scalarly by $\underset{\sim}{n}$ and using (3.10), we get

$$G_3(k_1 + k_2) = (\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta})(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{3.11}$$

<u>We now propose the following deterministic problem:</u> Let ξ and η be the surface coordinates on the surface ζ = constant, subject to the constraints

$$\Delta_2\xi = 0$$
$$\Delta_2\eta = 0 \tag{3.12}$$

Then the Cartesian coordinates x,y,z of the surface satisfy the differential equations

$$D\underset{\sim}{r} = G_3(k_1 + k_2)\underset{\sim}{n} \tag{3.13}$$

The three scalar differential equations for the generation of the Cartesian coordinates are then

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = XR \tag{3.14}$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = YR \tag{3.15}$$

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = ZR \tag{3.16}$$

where

$$R = (Xx_\zeta + Yy_\zeta + Zz_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{3.17}$$

and

$$\left.\begin{aligned} X &= (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3} \\ Y &= (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3} \\ Z &= (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3} \end{aligned}\right\} \tag{3.18}$$

Equations (3.14) - (3.16) form a quasilinear system of partial differential equations in which the components of $\underset{\sim}{r}_\zeta$ are assumed to be known. Since the values of x,y,z are known on the basic inner and outer boundaries (denoted at B and ∞ respectively in Fig. 1), a suitable way of prescribing $\underset{\sim}{r}_\zeta$ can be to take

$$\underset{\sim}{r}_\zeta = f_1(\eta)(\underset{\sim}{r}_\zeta)_B + f_2(\eta)(\underset{\sim}{r}_\zeta)_\infty \tag{3.19}$$

where $f_1(\eta)$ and $f_2(\eta)$ are suitable weights having the properties

$$f_1(\eta_B) = 1 \ , \ f_2(\eta_B) = 0$$

$$f_1(\eta_\infty) = 0 \ , \ f_2(\eta_\infty) = 1$$

For exposing the essential nonlinear terms in the factor R we refer to Eqs. (2.16) - (2.18) in which the $\underset{\sim}{r}_\zeta$ terms have been collected separately.

Referring to Figure 2, we now solve Eqs. (3.14) - (3.16) for each ζ = const., on a rectangular plane by prescribing the values of x, y and z on the lower side (C_1) and upper side (C_2) which represent the curves on B and ∞ respectively. The sides C_3 and C_4 are the cut lines on which periodic boundary conditions are to be imposed. The preceding analysis thus completes the formulation of the problem.

4. Coordinate Transformation (Contraction)

For the purpose of generating coordinates between the space of the inner and outer boundary which can be distributed in a desired manner, we consider a coordinate transformation from $\xi \to \chi$ and $\eta \to \sigma$. Let

$$\left.\begin{aligned} \xi &= \xi(\chi) + \xi_o \\ \eta &= \eta(\sigma) + \eta_B \end{aligned}\right. \tag{4.1}$$

then

$$\left.\begin{aligned} \xi &= \xi_o \text{ at } \chi = \chi_o \ , \ \xi(\chi_o) = 0 \\ \eta &= \eta_B \text{ at } \sigma = \sigma_B \ , \ \eta(\sigma_B) = 0 \end{aligned}\right. \tag{4.2}$$

Writing

$$\lambda(\chi) = \frac{d\xi}{d\chi} \ , \ \theta(\sigma) = \frac{d\eta}{d\sigma}$$

and denoting the transformed metric tensor as $\bar{g}_{ij}$, we have

$$\left.\begin{aligned} g_{11} &= \bar{g}_{11}/\lambda^2 \ , \ \bar{g}_{11} = x_\chi^2 + y_\chi^2 + z_\chi^2 \\ g_{12} &= \bar{g}_{12}/\theta\lambda \ , \ \bar{g}_{12} = x_\chi x_\sigma + y_\chi y_\sigma + z_\chi z_\sigma \\ g_{22} &= \bar{g}_{22}/\theta^2 \ , \ \bar{g}_{22} = x_\sigma^2 + y_\sigma^2 + z_\sigma^2 \\ G_3 &= \bar{G}_3/\theta^2\lambda^2 \ , \ \bar{G}_3 = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2 \\ X &= \bar{X} \ , \ Y = \bar{Y} \ , \ Z = \bar{Z} \end{aligned}\right\} \tag{4.3}$$

$$k_1 + k_2 = \bar{k}_1 + \bar{k}_2$$

$$R = \bar{R}/\theta^2\lambda^2$$

Further noting that

$$\left.\begin{aligned} \underset{\sim}{r}_{\xi\xi} &= (\underset{\sim}{r}_{\chi\chi} - \frac{\underset{\sim}{r}_\chi \lambda_\chi}{\lambda})/\lambda^2 \\ \underset{\sim}{r}_{\xi\eta} &= \underset{\sim}{r}_{\chi\sigma}/\theta\lambda \\ \underset{\sim}{r}_{\eta\eta} &= (\underset{\sim}{r}_{\sigma\sigma} - \frac{\underset{\sim}{r}_\sigma \theta_\sigma}{\sigma})/\theta^2 \end{aligned}\right\} \quad (4.4)$$

Using (4.3) and (4.4) in Eqs. (3.14) - (3.16), we have

$$\bar{g}_{22}x_{\chi\chi} - 2\bar{g}_{12}x_{\chi\sigma} + \bar{g}_{11}x_{\sigma\sigma} = Px_\chi + Qx_\sigma + \bar{X}\bar{R} \quad (4.5)$$

$$\bar{g}_{22}y_{\chi\chi} - 2\bar{g}_{12}y_{\chi\sigma} + \bar{g}_{11}y_{\sigma\sigma} = Py_\chi + Qy_\sigma + \bar{Y}\bar{R} \quad (4.6)$$

$$\bar{g}_{22}z_{\chi\chi} - 2\bar{g}_{12}z_{\chi\sigma} + \bar{g}_{11}z_{\sigma\sigma} = Pz_\chi + Qz_\sigma + \bar{Z}\bar{R} \quad (4.7)$$

where

$$\left.\begin{aligned} P &= \frac{\bar{g}_{22}}{\lambda}\lambda_\chi \\ Q &= \frac{\bar{g}_{11}}{\theta}\theta_\sigma \end{aligned}\right. \quad (4.8)$$

Thus, by choosing λ and θ arbitrarily we can redistribute the coordinates in the desired manner. An example of this choice is given in the next section.

5. An Analytical Example of Coordinate Generation

In this section we shall consider the problem of coordinate generation between a prolate ellipsoid and a sphere with coordinate contraction near the inner surface. This problem yields an exact solution of the equations (4.5) - (4.7).

Let $\eta = \eta_B$ and $\eta = \eta_\infty$ be the inner prolate ellipsoid and the outer sphere respectively. The coordinates which vary on these two surfaces are ξ and ζ. We now envisage a net of lines ξ = const. and ζ = const. on these two surfaces. A curve C_1 on the inner surface designated as $\zeta = \zeta_o$ is

$$\left.\begin{aligned} x &= \cosh\eta_B\cos\zeta_o \\ y &= \sinh\eta_B\sin\zeta_o\cos\xi \\ z &= \sinh\eta_B\sin\zeta_o\sin\xi \end{aligned}\right\} \qquad (5.1)$$

Similarly, the curve C_2 corresponding to $\zeta = \zeta_o$ on the outer surface is

$$\left.\begin{aligned} x &= e^{\eta_\infty}\cos\zeta_o \\ y &= e^{\eta_\infty}\sin\zeta_o\cos\xi \\ z &= e^{\eta_\infty}\sin\zeta_o\sin\xi \end{aligned}\right\} \qquad (5.2)$$

Based on the forms of the functions x,y,z in (5.1) and (5.2), we assume the following forms of x,y,z for the surface $\zeta = \zeta_o$:

$$\left.\begin{aligned} x &= f(\sigma)\cos\zeta_o \\ y &= \phi(\sigma)\sin\zeta_o\cos\xi \\ z &= \phi(\sigma)\sin\zeta_o\sin\xi \end{aligned}\right\} \qquad (5.3)$$

The boundary conditions for f and ϕ are

$$\left.\begin{aligned} f(\sigma_B) &= \cosh\eta_B \\ f(\sigma_\infty) &= e^{\eta_\infty} \\ \phi(\sigma_B) &= \sinh\eta_B \\ \phi(\sigma_\infty) &= e^{\eta_\infty} \end{aligned}\right\} \qquad (5.4)$$

Calculating the various derivatives, metric coefficients, and all other data needed in the equations (4.5) - (4.7), we get on substitution an equation which has $\sin^2\zeta_o$ and $\cos^2\zeta_o$. Equating to zero the coefficients of $\sin^2\zeta_o$ and $\cos^2\zeta_o$, we obtain

$$\frac{f''}{f'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.5)$$

$$\frac{\phi''}{\phi'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.6)$$

where a prime denotes differentiation with respect to σ. On direct integration of Eqs. (5.5) and (5.6) under the boundary conditions (5.4), we get

$$f(\sigma) = Ae^{B\eta(\sigma)} + C \tag{5.7}$$

$$\phi(\sigma) = De^{B\eta(\sigma)} \tag{5.8}$$

where

$$A = \frac{(e^{\eta_\infty} - \cosh\eta_B)\sinh\eta_B}{e^{\eta_\infty} - \sinh\eta_B} \tag{5.9a}$$

$$B = \ln[\frac{e^{\eta_\infty}}{\sinh\eta_B}]^{1/(\eta_\infty - \eta_B)} \tag{5.9b}$$

$$C = \frac{e^{\eta_\infty}(\cosh\eta_B - \sinh\eta_B)}{e^{\eta_\infty} - \sinh\eta_B} \tag{5.9c}$$

$$D = \sinh\eta_B \tag{5.9d}$$

As an application we may take [11]

$$\xi(\chi) = a\chi$$

$$\eta(\sigma) = b(\sigma - \sigma_B)K^{\sigma}$$

where a and b are constants. Since at η_∞,

$$\eta(\sigma_\infty) = \eta_\infty - \eta_B$$

hence

$$\eta(\sigma) = \frac{(\eta_\infty - \eta_B)(\sigma - \sigma_B)}{(\sigma_\infty - \sigma_B)}K^{(\sigma - \sigma_\infty)}$$

By taking a value of K slightly greater than one (K = 1.05 or 1.1), we can have sufficient contraction of coordinates near the inner surface.

For the chosen problem, since the dependence on ζ is simple, we find that the coordinates between a prolate ellipsoid and a sphere are

$$x = [Ae^{B\eta(\sigma)} + C]\cos\zeta$$

$$y = De^{B\eta(\sigma)}\sin\zeta\cos\xi$$

$$z = De^{B\eta(\sigma)}\sin\zeta\sin\xi$$

where A, B, C, and D are given in equation (5.9).

6. Conclusions

A new method for the generation of three-dimensional coordinates between two arbitrary shaped bodies has been presented. The method is based on some simple differential-geometric concepts such as the equations of Gauss and the expressions for the principal curvatures of a surface. The simplicity of the method lies in solving, at one time, only three partial differential equations of the two-dimensional type. This aspect is bound to reduce the working core requirements for a given problem on a computing machine. Finally the method allows, in a very direct fashion, the possibility of coordinate redistribution in the desired regions (cf. Eqs. (4.5) - (4.7)).

An analytic solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere has been presented. This example shows that one can generate coordinates between two analytically specified surfaces of simple forms by exact solutions of the proposed equations.

In this paper the fundamental equations which form a set of constraints for the generation of coordinates in the surface are

$$\Delta_2 \xi = 0$$

$$\Delta_2 \eta = 0 \ ,$$

where Δ_2 is the surface Beltrami operator of the second order. It must be noted that Δ_2 is neither a Laplace operator in the Cartesian plane (x,y), nor in the Cartesian space (x,y,z). However, in the case of a Cartesian plane (x,y), when there is no dependence on z, Δ_2 reduces to the Laplace operator ∇^2.

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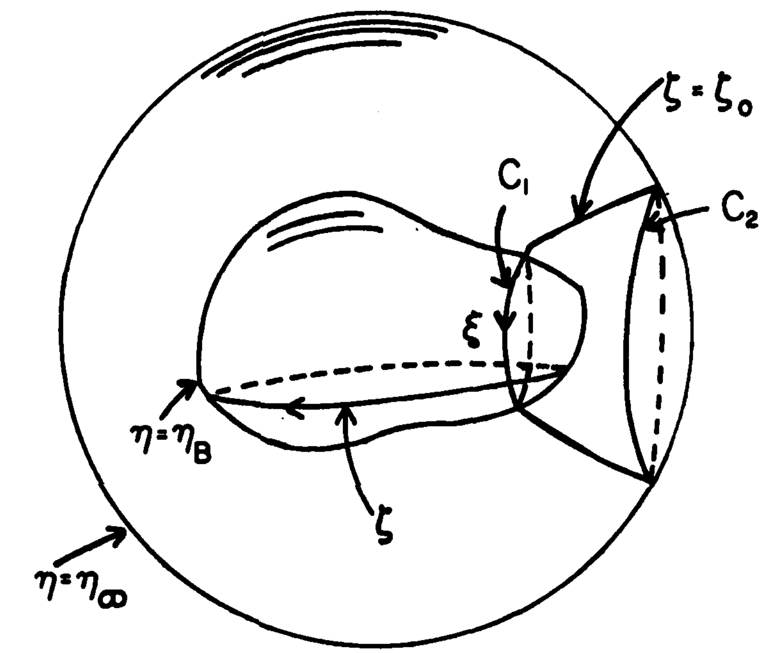
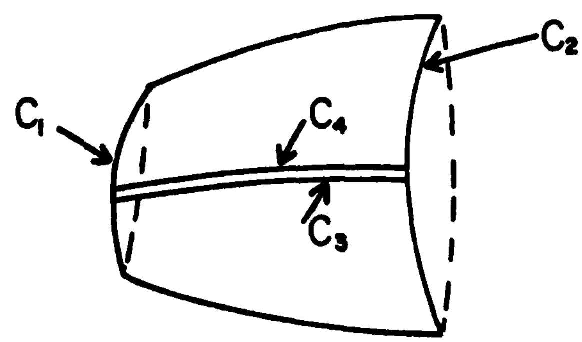


(a)



(b)

Figure 1: (a) Topoloty of the given surfaces. Inner $\eta = \eta_B$, outer $\eta = \eta_\infty$, current variables ξ, ζ. (b) Surface to be generated for each ζ = const., current variables ξ, η.

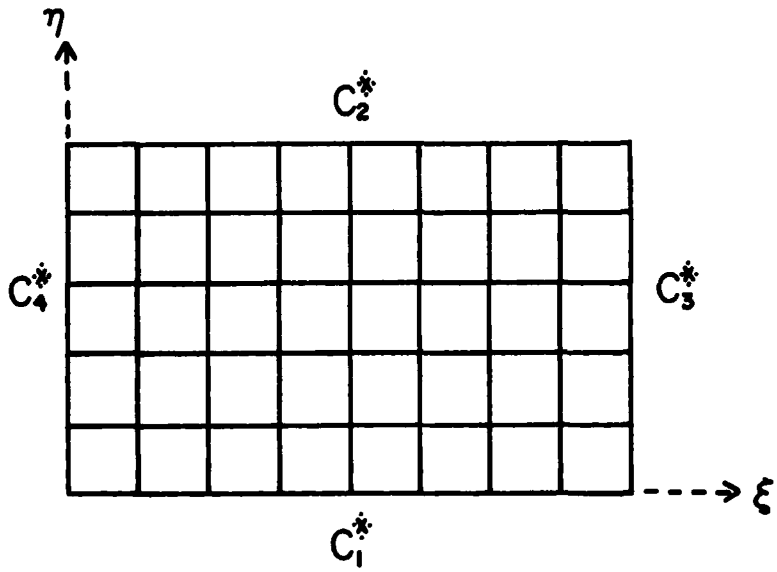


Figure 2: Figure 1(b) opened in a rectangular plane by imagining a cut.

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